TITLE: Constructive Approximation by [V,f]-Reproducing Kernels

Constructive Approximation by (V,f)-Reproducing Kernels

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Abstract. In this paper we propose a constructive method to build reproducing kernels. We define the notion of (V, f)-reproducing kernel, and prove that every reproducing kernel is a (V, f)-reproducing kernel. We study the minimal approximation by these (V, f)-reproducing kernels for different choices of V and f. Examples to which our results apply include curve and surface fitting.

§1. (V,f)-Reproducing Kernels

For any set (respectively locally compact set) Ω, we denote by $\mathbb{R}^\Omega$ (respectively $\mathcal{C}^m(\Omega)$) the space of real-valued functions (respectively m-times continuously differentiable functions) defined on Ω equipped with the topology of pointwise convergence (respectively uniform convergence on the compact subsets of Ω). Let us recall some definitions.

Definition 1.1. *A real-valued function H defined on $\Omega \times \Omega$ is a* reproducing kernel *on $\Omega \times \Omega$ if*

1) *H is symmetric: $H(t,s) = H(s,t)$ for all t, $s \in \Omega$,*
2) *H is of positive type:*

$$\sum_{k,l=1}^{k,l=N} \lambda_k \lambda_l H(t_k, t_l) \geq 0,$$

for any finite point set $\{t_k\}_{k=1}^N$ of Ω and real numbers $\{\lambda_k\}_{k=1}^N$.

Definition 1.2. *A vector subspace $\mathcal{H}$ of $\mathbb{R}^\Omega$ is said to be a* hilbertian subspace *of $\mathbb{R}^\Omega$ (respectively $\mathcal{C}^m(\Omega)$) if*

1) *$\mathcal{H}$ is a Hilbert space,*
2) *The natural injection from $\mathcal{H}$ into $\mathbb{R}^\Omega$ (respectively $\mathcal{C}^m(\Omega)$) is continuous.*

We review some important results on reproducing kernels which are studied in [4].

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Theorem 1.1.

1) *A Hilbert space $\mathcal{H}$ (respectively a real-valued function H defined on $\Omega\times\Omega$) is a hilbertian subspace of $\mathbb{R}^\Omega$ (respectively a reproducing kernel on $\Omega\times\Omega$) if and only if there exists one and only one reproducing kernel H on $\Omega\times\Omega$ (respectively hilbertian subspace $\mathcal{H}$ of $\mathbb{R}^\Omega$) such that*

$$u(t) = \langle u \mid H(\cdot,t)\rangle_{\mathcal{H}}, \qquad \forall t \in \Omega, \quad \forall u \in \mathcal{H}.$$

$\mathcal{H}$ is called the hilbertian subspace *associated with H.*

2) *For any hilbertian basis $(f_i)_{i\in I}$ of $\mathcal{H}$: $H(t,s) = \sum_{i\in I} f_i(t)f_i(s)$.*

3) *If H is separately m-times continuously differentiable, then $\mathcal{H}$ is a hilbertian subspace of $C^m(\Omega)$.*

4) *The vector space $\mathcal{H}_0$ =span$\{\big(H(\cdot,t)\big)_{t\in\Omega}\}$ is dense in $\mathcal{H}$.*

Let $(V, \langle\cdot \mid \cdot\rangle_V)$ be a Hilbert space, Ω be a set and f be a function from Ω into V.

Definition 1.3. *For all $f : \Omega \longrightarrow V$, we define a (V, f)*-reproducing kernel *H_f by*

$$H_f(t,s) = \langle f(t) \mid f(s)\rangle_V, \qquad \forall (t,s) \in \Omega\times\Omega. \tag{1.1}$$

We have the following result:

Theorem 1.2. *H_f defined by (1.1) is a reproducing kernel on $\Omega\times\Omega$ and its associated hilbertian subspace $\mathcal{H}_f$ of $\mathbb{R}^\Omega$ is*

$$\mathcal{H}_f = \Big\{w \in \mathbb{R}^\Omega \mid \exists u \in V: \ w(t) = \langle u \mid f(t)\rangle_V, \ \forall t \in \Omega\Big\}.$$

Proof: One can easily verify that H_f is a reproducing kernel.
Let $\tilde{H}_f : V \longrightarrow \mathbb{R}^\Omega$ be defined by $(\tilde{H}_f u)(t) = \langle u \mid f(t)\rangle_V$. The mapping $\tilde{H}_f$ is linear, and the inequality

$$\mid (\tilde{H}_f u)(t) \mid \leq \mid u \mid_V \mid f(t) \mid_V = \mid u \mid_V \ H_f(t,t)^{\frac{1}{2}},$$

for all $t \in \Omega$ and for all $u \in V$ implies that it is continuous. Let $\mathcal{M}$ be the closure in V of the vector space span$\{\big(f(t)\big)_{t\in\Omega}\}$, and $P_{\mathcal{M}}$ the orthogonal projector on $\mathcal{M}$. We define on $\mathcal{H}_f = \tilde{H}_f(V)$ the bilinear form

$$\langle \tilde{H}_f u \mid \tilde{H}_f v\rangle_{\mathcal{H}_f} = \langle P_{\mathcal{M}} u \mid P_{\mathcal{M}} v\rangle_V.$$

It is easy to see that this form is a scalar product on $\mathcal{H}_f$. Then the linear mapping $\tilde{H}_f : \mathcal{M} \longrightarrow \mathcal{H}_f$ is an isometry, and consequently $(\mathcal{H}_f, \langle\cdot \mid \cdot\rangle_{\mathcal{H}_f})$ is a Hilbert space. For all $t \in \Omega$, the function

$$H_f(t,\cdot) : s \in \Omega \longrightarrow H_f(t,s) = \langle f(t) \mid f(s) \rangle_V,$$

is an element of $\mathcal{H}_f$, and satisfies the reproducing formula

$$(\tilde{H}_f u)(t) = \langle \tilde{H}_f u \mid H_f(t,\cdot) \rangle_{\mathcal{H}_f}, \qquad \forall u \in V.$$

Consequently (see Theorem 1.1), $\mathcal{H}_f$ is a hilbertian subspace of $\mathbb{R}^\Omega$ and admits H_f as reproducing kernel. □

Theorem 1.3. *Let ω be a set and c a mapping from ω to Ω. Then*

$$(Hc)_f(y,z) = H_f(c(y),c(z)) = \langle f(c(y)) \mid f(c(z)) \rangle_V$$

is a reproducing kernel on $\omega \times \omega$.

Proof: For all $y \in \omega$, $f(c(y))$ is in V. The function $(Hc)_f$ is symmetric and is of positive type:

$$\sum_{k,l=1}^{k,l=N} \lambda_k \lambda_l (Hc)_f(y_k, y_l) = \langle \sum_{k=1}^{k=N} \lambda_k f(c(y_k)) \mid \sum_{k=1}^{k=N} \lambda_k f(c(y_k)) \rangle_V \geq 0. \ \square$$

Example 1.1. Let $V = L^2(a,b)$, Ω a subset of $\mathbb{R}$ and $f(t)(x) = \exp(cxt)$ where c is a real constant. Then

$$H_f(t,s) = \begin{cases} (\exp(cb(t+s)) - \exp(ca(t+s)))/(c(t+s)), & \text{if } (t+s) \neq 0, \\ b-a, & \text{otherwise.} \end{cases}$$

Example 1.2. Let $V = L^2(\mathbb{R}^+)$, and suppose Ω is a subset of $\mathbb{R}^n$.
For all functions $c : \Omega \longrightarrow (0,+\infty)$, we have

(i) If $f(t)(x) = \frac{2}{\pi^{\frac{1}{4}}} \exp^{-c(t)|x|^2}$, then $H_f(t,s) = \dfrac{1}{\sqrt{c(t)+c(s)}}$.

(ii) If $f(t)(x) = \exp^{-c(t)x}$, then $H_f(t,s) = \dfrac{1}{c(t)+c(s)}$, and in particular if $c(t) = \dfrac{P(t)}{Q(t)}$ (with $P(t)$ and $Q(t)$ polynomials), we obtain the rational reproducing kernel

$$H_f(t,s) = \frac{Q(t)Q(s)}{P(t)Q(s) + P(s)Q(t)}.$$

§2. (V,f)-Reproducing Kernels of Convolution Type

We consider the case where

1) $V = L^2(\mathbb{R}^n)$ and $\Omega = \mathbb{R}^n$.

2) $f(t)(x) = f(t-x)$, with f in the familar Sobolev space $H^m(\mathbb{R}^n)$.

Then $H_f(t,s) = \displaystyle\int_{\mathbb{R}^n} f(t-x)f(s-x)dx$.

Theorem 2.1. *We have the following properties:*

1) $H_f(t,s) = h_f(t-s)$ *with* $h_f(\xi) = (f * \check{f})(\xi) = \mathcal{F}(\mid \mathcal{F}f \mid^2)(\xi)$, *where* $\check{f}(x) = f(-x)$ *and* $\mathcal{F}f$ *is the Fourier transform of* f.
2) $h_f \in \mathcal{C}_0^m(\mathbb{R}^n) = \left\{u \in \mathcal{C}^m(\mathbb{R}^n) \mid \lim\limits_{|t|\to\infty} (D^\alpha u)(t) = 0, \quad 0 \leq \mid \alpha \mid \leq m\right\}$.
3) *The associated hilbertian subspace of* H_f *is*

$$\mathcal{H}_f = f * L^2(\mathbb{R}^n) \quad \hookrightarrow \mathcal{C}_0^m(\mathbb{R}^n) \text{ (continuous embedding)}.$$

4) *In particular, if* $\mid \mathcal{F}f \mid > 0$, *then*

$$\mathcal{H}_f = \left\{w \in \mathcal{S}' \mid \mathcal{F}w \in L^1_{loc}(\mathbb{R}^n), \quad \frac{\mathcal{F}w}{\mathcal{F}f} \in L^2(\mathbb{R}^n)\right\},$$

equipped with the scalar product

$$\langle w_1 \mid w_2 \rangle_{\mathcal{H}_f} = \int_{\mathbb{R}^n} \frac{\mathcal{F}w_1(\xi)\,\overline{\mathcal{F}w_2(\xi)}}{\mid \mathcal{F}f(\xi) \mid^2} d\xi.$$

5) If f *is radial, then* h_f *is radial:* $H_f(t,s) = h_f(\mid t-s \mid)$.
6) For all distinct points $\{t_k\}_{k=1}^N$ *in* $\mathbb{R}^n$, *the matrix* $H_N = \left(H_f(t_k, t_l)\right)_{1\leq k,l \leq N}$ *is invertible (strictly positive definite).*

Proof:
1) We have

$$H_f(t,s) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-i\langle t-s|\xi\rangle} \mid \mathcal{F}f(\xi)|^2 d\xi = \mathcal{F}(\mid \mathcal{F}f \mid^2)(t-s).$$

2) $f \in H^m(\mathbb{R}^n) \Rightarrow D^\alpha h_f = (D^\alpha f) * \check{f} \in \mathcal{C}_0^0(\mathbb{R}^n)$ for $0 \leq \mid \alpha \mid \leq m$, (see [2]).
3) is a consequence of Theorem 1.2 and the property given in 1).
4) Since $\mathcal{H}_f \hookrightarrow \mathcal{C}_0^m(\mathbb{R}^n) \hookrightarrow \mathcal{S}'$, we have the equivalences:

$$\begin{aligned}\{w \in \mathcal{H}_f\} &\Leftrightarrow \{\exists u \in L^2(\mathbb{R}^n) : \mathcal{F}w = \mathcal{F}u\mathcal{F}f\} \\ &\Leftrightarrow \{\mathcal{F}w \in L^1_{loc}(\mathbb{R}^n), \frac{\mathcal{F}w}{\mathcal{F}f} \in L^2(\mathbb{R}^n)\}.\end{aligned}$$

From Theorem 1.2, we have

$$w(t) = \langle u \mid f(t-\cdot)\rangle_{L^2(\mathbb{R}^n)} = \langle P_{\mathcal{M}}u \mid f(t-\cdot)\rangle_{L^2(\mathbb{R}^n)} = (P_{\mathcal{M}}u * f)(t).$$

Then $\mathcal{F}w = \mathcal{F}P_{\mathcal{M}}\, u\mathcal{F}f$, and

$$\begin{aligned}\langle w_1 \mid w_2 \rangle_{\mathcal{H}_f} &= \int_{\mathbb{R}^n} P_{\mathcal{M}}u_1(x) P_{\mathcal{M}}u_2(x) dx \\ &= \int_{\mathbb{R}^n} \mathcal{F}P_{\mathcal{M}}u_1(\xi)\,\overline{\mathcal{F}P_{\mathcal{M}}u_2(\xi)} d\xi = \int_{\mathbb{R}^n} \frac{\mathcal{F}w_1(\xi)\,\overline{\mathcal{F}w_2(\xi)}}{\mid \mathcal{F}f(\xi) \mid^2} d\xi.\end{aligned}$$

5) For any orthogonal matrix A,

$$h_f(At) = \int_{\mathbf{R}^n} f(x)f(At-x)dx = \int_{\mathbf{R}^n} f(Ax)f(A(t-x))dx$$
$$= \int_{\mathbf{R}^n} f(x)f(t-x)dx = h_f(t),$$

since $f(Ax) = f(x)$ and $\mid detA \mid = 1$.

6) We suppose that $f \not\equiv 0$ in $L^2(\mathbb{R}^n)$. Since the matrix $H_N = \left(\int_{\mathbf{R}^n} f(t_k - x)f(t_l - x)dx\right)_{1\le k,l\le N}$ is a Gram matrix, it is invertible if and only if the system $\{f(t_k - \cdot)\}_{k=1}^{N}$ is lineary independent in $L^2(\mathbb{R}^n)$. If $\sum_{k=1}^{k=N} c_k f(t_k - x) = 0$ in $L^2(\mathbb{R}^n)$, for $c_k \in \mathbb{R}$, $1 \le k \le N$, then by the Fourier transform we get $\left(\sum_{k=1}^{k=N} c_k e^{-i\langle t_k|\xi\rangle}\right)\mathcal{F}f(\xi) = 0$ in $L^2(\mathbb{R}^n)$.

The Lebesgue measure of the set $\mathcal{N} = \{\xi \in \mathbb{R}^n \mid \sum_{k=1}^{k=N} c_k e^{-i\langle t_k|\xi\rangle} = 0\}$ is equal to zero. Then $\mathcal{F}f$ vanishes outside $\mathcal{N}$, i.e: $\mathcal{F}f \equiv 0$ in $L^2(\mathbb{R}^n)$ and by the inverse Fourier transform, $f \equiv 0$ in $L^2(\mathbb{R}^n)$, which complete the proof. □

Example 2.1. Let $u(x) = (1- \mid x \mid)_+$ and $\mathcal{F}u = v$. We have $v(x) = \dfrac{sin^2(\frac{x}{2})}{x^2}$.

(i) Taking $f = \mathcal{F}(\mid u \mid^{\frac{1}{2}})$, $H_f(t,s) = (1- \mid t-s \mid)_+$.

(ii) Taking $f = \mathcal{F}(\mid v \mid^{\frac{1}{2}})$, $H_f(t,s) = \dfrac{sin^2(\frac{t-s}{2})}{(t-s)^2}$.

Example 2.2. (Bessel reproducing kernels) For $n \in \mathbb{N}$, $\alpha \in \mathbb{R}$ and $\alpha > n$; consider $G_\alpha \in L^2(\mathbb{R}^n)$ defined by $\mathcal{F}(G_\alpha)(x) = (1 + \mid x \mid^2)^{-\frac{\alpha}{2}}$.

(i) Taking $f = \mathcal{F}(\mid G_\alpha \mid^{\frac{1}{2}})$, $H_f(t,s) = (1+ \mid t-s \mid^2)^{-\frac{\alpha}{2}}$.

(ii) Taking $f = \mathcal{F}(\mid \mathcal{F}G_{n+1} \mid^{\frac{1}{2}})$, $H_f(t,s) = \frac{\pi^{\frac{1-n}{2}}}{2^n\Gamma(\frac{n+1}{2})}\exp(-\mid t-s \mid)$ and $\mathcal{H}_f = H^{\frac{n+1}{2}}(\mathbb{R}^n)$ (Sobolev space).

Example 2.3. (ν-B-spline reproducing kernels) Let

1) $Y_l(x) = \frac{1}{l!}x_+^l$.
2) $\nu \in \mathcal{E}'$ (distributions with compact support) such that $\nu(p) = 0$, for all polynomial p in $\mathcal{P}_l(\mathbb{R})$.
3) $f = \nu * Y_l$.

For such functions f, we give the following theorem without proof.

Theorem 2.2. *We have:*

1) $f \in L^2(\mathbb{R})$.
2) *For all* u *in* $V^{l+1}(\mathbb{R}) = \{v \in L^2_{loc}(\mathbb{R}) / \ v^{(l+1)} \in L^2(\mathbb{R})\}$ *(Beppo-Levi space) we have* $\int_{\mathbb{R}} u^{(l+1)}(x) f(t-x)dx = (\nu * u)(t)$.
3) $H_f(t,s) = (-1)^l (\check{\nu} * \nu * Y_{2l+1})(t-s)$ *and* $\mathcal{H}_f = \nu * V^{l+1}(\mathbb{R})$.

In the particular case of divided differences, ν is defined as the mth-iterated convolution $\nu = \frac{(\delta_a - \delta_b)^{*m}}{(b-a)}$, and $\check{\nu} = \frac{(\delta_{-a} - \delta_{-b})^{*m}}{(b-a)}$.

§3. Data Fitting by (V,f)-Reproducing Kernels

Let $\{t_k\}_{k=1}^N$ a set of distinct points in Ω, and define a linear operator A_N from $\mathcal{H}_f$ into $\mathbb{R}^N$ by $A_N(u) = \big(u(t_k)\big)_{1 \le k \le N}$.

Definition 3.1. *For all* $z_N \in \mathbb{R}^N$ *and* $\epsilon \in [0,1)$ *we define a spline to be any solution of the following minimal approximation problem:*

$$\Big(P_\epsilon(z_N)\Big): \qquad \inf_{u \in C_\epsilon} \Big((1-\epsilon)\langle u \mid u\rangle_{\mathcal{H}_f} + \epsilon \|A_N u - z_N\|^2_{\mathbb{R}^N}\Big),$$

where

$$C_\epsilon = \begin{cases} A_N^{-1}\{z_N\}, & \text{if } \epsilon = 0 \ (\textit{Interpolation}), \\ \mathcal{H}_f, & \text{if } \epsilon \in]0,1[\ (\textit{Smoothing}). \end{cases}$$

The following theorem gives the spline in the case $\epsilon \neq 0$.

Theorem 3.1. *For all* $(\epsilon, z_N) \in]0,1[\times \mathbb{R}^N$, *the problem* $P_\epsilon(z_N)$ *(Smoothing) admits a unique solution*

$$\sigma^\epsilon(t) = \sum_{k=1}^{k=N} \lambda_k^\epsilon H_f(t, t_k),$$

where the coefficients ${}^t\Lambda^\epsilon = (\lambda_1^\epsilon, \dots, \lambda_N^\epsilon) \in \mathbb{R}^N$ *are the solution of the system*

$$(H_N + \frac{\epsilon}{1-\epsilon} I_N)\Lambda_N^\epsilon = z_N,$$

with $H_N = (H_f(t_k, t_l))_{1 \le k,l \le N}$ *and* I_N *is the identity matrix.*

Proof: 1) From the continuous embedding: $\mathcal{H}_f \hookrightarrow \mathbb{R}^\Omega$ (see Theorem 1.2), we deduce that A_N is continuous. 2) $A_N(\mathcal{H}_f)$ is closed as a vector subspace of $\mathbb{R}^N$. Then from the general spline theory (see [1,3]) we get the theorem. □

Theorem 3.2. *The following two properties are equivalent:*
1) For all $z_N \in \mathbb{R}^N$, the problem $P_0(z_N)$ admits a unique solution.
2) The system $\{f(t_k)\}_{k=1}^N$ is linearly independent in V.

Proof: For all $z_N \in \mathbb{R}^N$, the problem $P_0(z_N)$ admits a unique solution if and only if the matrix $H_N = \big(H_f(t_k, t_l)\big)_{1\leq k,l\leq N}$ is invertible (see [1,3]). Since the matrix $H_N = \big(\langle f(t_k) \mid f(t_l)\rangle_V\big)_{1\leq k,l\leq N}$ is a Gram matrix, it is invertible if and only if the system $\{f(t_k)\}_{k=1}^N$ is lineary independent in V. □

Furthermore, for the particular case $V = L^2(\mathbb{R}^n)$ and $f \in H^m(\mathbb{R}^n)$, Theorem 3.2 and the property (6) of Theorem 2.1 imply the following theorem.

Theorem 3.3. *For all $f \in H^m(\mathbb{R}^n)$ and $z_N \in \mathbb{R}^N$, the problem $P_0(z_N)$ (Interpolation) admits a unique solution*

$$\sigma^0(t) = \sum_{k=1}^{k=N} \lambda_k^0 H_f(t, t_k),$$

where the coefficients ${}^t\Lambda^0 = (\lambda_1^0, \ldots, \lambda_N^0) \in \mathbb{R}^N$ are the solution of the system

$$H_N \Lambda_N^0 = z_N,$$

with $H_N = (H_f(t_k, t_l))_{1\leq k,l\leq N}$ and I_N is the identity matrix.

§4. Data Fitting Preserving Polynomials

Let $\mathcal{P}_d(\mathbb{R}^n)$ the vector space of polynomials of degree at most d. We suppose:

(H1) For all $p \in \mathcal{P}_d(\mathbb{R}^n)$ the subset $\{t_k\}_{k=1}^N$ of $\mathbb{R}^n$ is such that

$$\{p(t_k) = 0, \quad 1 \leq k \leq N\} \Longleftrightarrow p \equiv 0.$$

(H2) $\mathcal{H}_f \cap \mathcal{P}_d(\mathbb{R}^n) = \{0\}$.

We remark that in the case $V = L^2(\mathbb{R}^n)$ and $f \in H^m(\mathbb{R}^n)$, the hypothesis (H2) is satisfied because $\mathcal{H}_f \subset C_0^m(\mathbb{R}^n)$ (see Theorem 2.1(2)), and

$$C_0^m(\mathbb{R}^n) \cap \mathcal{P}_d(\mathbb{R}^n) = \{0\}.$$

Let $\mathcal{H}_f^d$ be the Hilbert direct sum: $\mathcal{H}_f^d = \mathcal{H}_f \oplus \mathcal{P}_d(\mathbb{R}^n)$. We denote by Π_f the orthogonal projector from $\mathcal{H}_f^d$ onto $\mathcal{H}_f$, and we define on $\mathcal{H}_f^d$ the linear mapping $A_N(u) = \big(u(t_k)\big)_{1\leq k\leq N} \in \mathbb{R}^N$. For all $(\epsilon, z_N) \in [0,1] \times \mathbb{R}^N$, we consider the following minimal approximation problem in $\mathcal{H}_f^d$:

$$\Big(P_\epsilon(z_N)\Big) : \qquad \inf_{u\in C_\epsilon} \Big((1-\epsilon)\langle \Pi_f(u) \mid \Pi_f(u)\rangle_{\mathcal{H}_f} + \epsilon \|A_N u - z_N\|^2_{\mathbb{R}^N}\Big),$$

where

$$C_\epsilon = \begin{cases} A_N^{-1}\{z_N\}, & \text{if } \epsilon = 0 \text{ (Interpolation)}, \\ \mathcal{H}_f^d, & \text{if } \epsilon \in]0,1[\text{ (Smoothing)}, \\ \mathcal{P}_d(\mathbb{R}^n), & \text{if } \epsilon = 1 \text{ (Least squares)}. \end{cases}$$

The hypothesis (H1) implies that the problem $P_1(z_N)$ admits a unique solution. In the case $\epsilon \in]0,1[$ (Smoothing) we have the following theorem:

Theorem 4.1. *For all* $(\epsilon, z_N) \in]0,1[\times \mathbb{R}^N$, *the problem* $P_\epsilon(z_N)$ *admits a unique solution* σ^ϵ. *In the case* $\epsilon \in]0,1[$, *the solution* σ^ϵ *is given by*

$$\sigma^\epsilon(t) = \sum_{k=1}^{k=N} \lambda_k^\epsilon H_f(t, t_k) + \sum_{i=1}^{i=n_d} b_i^\epsilon p_i(t),$$

where the coefficients ${}^t\Lambda^\epsilon = (\lambda_1^\epsilon, \ldots, \lambda_N^\epsilon) \in \mathbb{R}^N$ *and* ${}^tB^\epsilon = (b_1^\epsilon, \ldots, b_{n_d}^\epsilon) \in \mathbb{R}^{n_d}$ *are the solution of the system*

$$\begin{pmatrix} H_N + \frac{\epsilon - 1}{\epsilon} I_N & E \\ {}^tE & 0 \end{pmatrix} \begin{pmatrix} \Lambda^\epsilon \\ B^\epsilon \end{pmatrix} = \begin{pmatrix} z_N \\ 0 \end{pmatrix},$$

with

1) $H_N = \left(H_f(t_k, t_l)\right)_{1 \le k,l \le N}$ *and* I_N *is the identity matrix,*

2) $E = \left(E_{k,i}\right)_{1 \le i \le n_d}^{1 \le k \le N}$ *with* $E_{k,i} = p_i(t_k)$ *and* $(p_i)_{1 \le i \le n_d}$ *is a basis of* $\mathcal{P}_d(\mathbb{R}^n)$.

In particular, if there exists $p \in \mathcal{P}_d(\mathbb{R}^n)$ *such that* $\{p(t_k) = z_{N,k}, 1 \le k \le N\}$, *then* $\sigma^\epsilon = p$ *(preserving polynomials property).*

Proof: Theorem 4.1 is a consequence of general spline theory (see [1,3]):

1) A_N is continuous since $\mathcal{H}_f^d$ is a hilbertian subspace of $\mathbb{R}^\Omega$.

2) Π_f is continuous and $\Pi_f(\mathcal{H}_f^d) = \mathcal{H}_f$ is closed since Π_f is an orthogonal projector.

3) $\ker A_N \cap \ker \Pi_f = \{0\}$: derives from the hypothesis (H1) and the fact that $\ker \Pi_f = \mathcal{P}_d(\mathbb{R}^n)$.

4) $\ker A_N + \ker \Pi_f$ is closed since $\ker \Pi_f$ is a finite dimensional vector space.

$\square$

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